12. Hydraulic jumps and Bores

— A tsunami wave travelling upstream along a river

I. Classical Discussion on Hydraulic Jumps and Bores

1. Hydraulic Jump

After passing over a dam, the stream makes a "rolling tube" of water downstream. We call such features a "hydraulic jump."

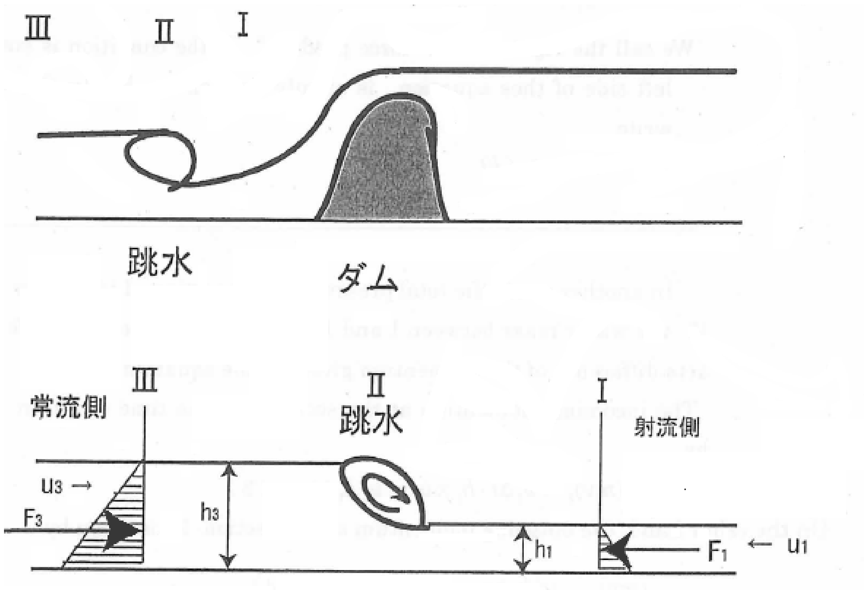

Fig. 1 Hydraulic Jump

In figure 1, section I is considered to be before the hydraulic jump, and section III after it. The status of section I is "supercritical flow" or "shoot," and that of section III is "subcritical flow." We assume that water depths at I and III are h_1 and h_3, and the average flow speeds are u_1 and u_3, respectively. We assume that the location of the hydraulic jump does not change, and the 跳水の位 flow is stationary. The law of conservation of mass must be satisfied, that is, the incoming flow amount at section I is equal that of outgoing flow at the section III, so,

$$u_1 h_1 = u_3 h_3,$$

We call this value the "flow amount," q

$$u_1 h_1 = u_3 h_3 \equiv q \quad (1)$$

We next consider the momentum of the water mass between the sections I and III. We compare the incoming momentum passing though section I, and the outgoing momentum passing through III; the former is larger than the latter. The difference of the momentums is derived from the principle that “the momentum change is equal to the product of force and duration time ($\Delta mv = Ft$).”

Note: If we integrate $F = m\alpha$ by time, then we have

$$\int_A^B F dt = [mv]_{t_A}^{t_B}$$

We call the right side the “force product.” If the condition is stationary, the left side of this equation is simply written as *Ft*. In this case, we can write

$$\Delta(mv) = Ft \qquad (2)$$

In other words, the total pressure at section I is less than that at III; the water mass between I and III receives this pressure difference, which acts as the difference in momentum given by equation (2).

The incoming momentum at section I over time Δt is given by

$$(mv)_1 = u_1 \Delta t \cdot h_1 \times u_1 = u_1^2 h_1 \Delta t \qquad (3)$$

However, the outgoing momentum at section III is given by

$$(mv)_3 = u_3^2 h_3 \Delta t \qquad (4)$$

The difference in momentum between I and III is given by

$$\rho\left(u_1^2 h_1 - u_3^2 h_3\right)\Delta t \qquad (5)$$

Per unit width. This difference is equal to the force product. The total pressure at the section I is

$$F_1 = \int_0^{h_1} p dz = \int_0^{h_1} \rho(h_1 - z) g dz = \frac{1}{2}\rho g h_1^2 \qquad (6)$$

in the same way that the total pressure at section III is

$$F_3 = \frac{1}{2}\rho g h_3^2 \qquad (7)$$

As $h_3 > h_1$, so $F_3 > F_1$. This difference of the total force causes the force product in the duration time Δt. So,

$$(F_3 - F_1)\Delta t = \rho(u_1^2 h_1 - u_3^2 h_3) \qquad (8)$$

Or

$$\frac{1}{2} g\left(h_3^2 - h_1^2\right) = \left(u_1^2 h_1 - u_3^2 h_3\right) \qquad (9)$$

Here, the right side may be written as

$$q^2(1/h_1 - 1/h_3) = q^2\left(\frac{h_3 - h_1}{h_1 h_3}\right)$$

Thus, equation (9) is

$$\frac{g}{2}(h_1 + h_3) = \frac{q^2}{h_1 h_3} \qquad (10)$$

If we consider h_1, u_1, and q as the known numbers, and h_3 as an unknown number, we have the following equation of the second order,

$$h_3^2 + h_1 h_3 - \frac{2q^2}{g h_1} = 0$$

The solution of this equation is

$$h_3 = -\frac{h_1}{2} + \sqrt{\frac{h_1^2}{4} + \frac{2q^2}{g h_1}} = -\frac{h_1}{2} + \sqrt{\frac{h_1^2}{4} + \frac{2 h_1 u_1^2}{g}} \qquad (11)$$

Thus, we have a value for h_3, and can easily obtain the value of u_3 using equation (1).

2. Energy Change before and after a Hydraulic Jump

Bernoulli's equation for a stationary stream is given by

$$\frac{1}{2}\left(u^2 + v^2 + w^2\right) + gz + p = Const. \qquad (12)$$

We can recognize that equation (12), when multiplied by water density ρ, expresses energy contained in a unit volume. The first term corresponds to kinematic energy, $1/2 \cdot mv^2$, and the second term mgh is the potential energy. It should be noted that the energy contained in a unit volume near the surface, at middle depth, or near the bottom are the same, because the p-value will compensate for the decrease of the second term value.

At a point on the water surface in section I, the energy per unit volume is given by

$$\left(\frac{1}{2}u_1^{\,2}+g\cdot 0\right)+\frac{u_1^{\,2}}{2}$$

Thus, the energy passing section I per unit time is expressed as,

$$E_1=\frac{q}{2}u_1^{\,2}$$

In the same way, the energy per unit volume at section III is

$$\frac{1}{2}u_3^{\,2}+g(h_3-h_1)$$

Thus, the total energy going out through section III per unit time is given by

$$E_3=q\left(\frac{1}{2}u_3^{\,2}+g(h_3-h_1)\right)$$

We take the difference, and then, we have

$$\Delta E=E_1-E_3=q\left(\frac{1}{2}\left(u_1^{\,2}-u_3^{\,2}\right)-g(h_3-h_1)\right) \qquad (13)$$

This equation can be re-written by using equation (10)

$$g=\frac{2q^2}{h_1h_3(h_1+h_3)},\ \ u_1=\frac{q}{h_1},\ \ u_3=\frac{q}{h_3}$$

Equation (13) turns into

$$\Delta E=\frac{q^3(h_3-h_1)^3}{2h_1h_3(h_1+h_3)} \qquad (14)$$

As $h_3 > h_1$ the value of this will be a positive number for any case. That is, in every case, some part of the total energy will be missing. The ratio of the depths before and after the hydraulic jump is expressed as $s = h_3/h_1$, and then, this equation becomes

$$\Delta E=\frac{q^3(s-1)^3}{2s(s+1)}$$

If s is close to 1, this value is very small (weak jump), while if s is far from 1 (strong jump), the energy dissipation becomes larger.

3. Subcritical flow and Supercritical flow

If we have the flow amount (q) and the flow velocity (u), we may calculate the energy in a unit volume (E). The depth h is related to the flow amount and velocity as

$$u = q/h$$

The energy per unit volume is

$$E = \frac{1}{2}u^2 + gh \qquad (15)$$

By using equation (14), we eliminate u and have

$$E = \frac{1q^2}{2h^2} + gh \qquad (16)$$

This value has a minimum of (we apply $(a + b + c)/3 \geq \sqrt[3]{abc};$)

$$E \geq \frac{3}{2}\sqrt[3]{g^2q^2}$$

We find that

$$q^2/h^2 = gh$$

In the case when

$$u = \sqrt{gh} \qquad (17)$$

The right side gives the long wave speed. This means that in the case when flow is equal to the long wave speed, the energy is minimized.

In the case that the flow speed is slower than the speed of a long wave, water surface disturbances can go upstream, and a disturbance of the water surface appears then is rapidly recovered, and it is possible to keep the water surface flat like a mirror. We call such flow as "sub-critical flow."

In contrast, in the case that the flow speed exceeds the long wave speed, disturbances appear on the water surface and cannot go upstream, and so they cannot be flattened, and the water surface disturbance cannot be recovered. We call such flow as "supercritical flow" or "shoot."

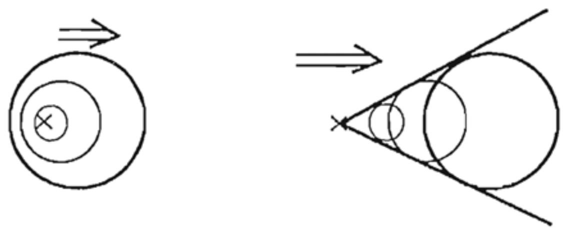

Fig. 2

$$u = \sqrt{gh} \qquad (18)$$

The ratio of the flow speed to the long wave speed is known as the Frued Number (Fr).

$$Fr \equiv u/\sqrt{gh} \qquad (19)$$

In the case where the Fr number does not exceed 1, subcritical flow occurs, and in contrast, supercritical flow occurs when the flow speed exceeds the long wave speed.

4. Bore

In the case when a tsunami wave comes into a river, the water surface takes a step-shaped-form called a "bore."

Essentially, a bore is a moving hydraulic jump.

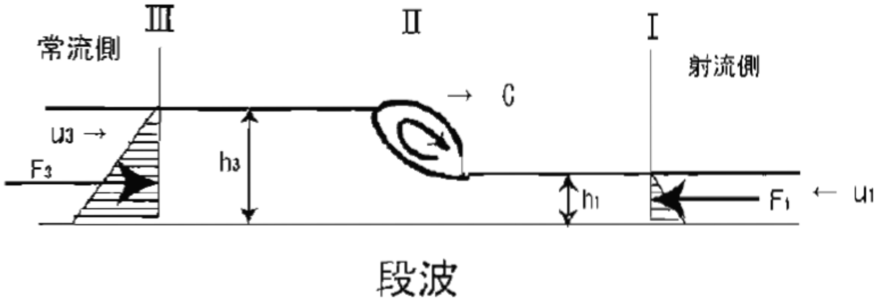

Fig. 3. Bore

This phenomenon may be observed in a driving car that has the same speed as the bore speed; the bore does not appear to be moving, and the observation matter is the same as a hydraulic jump. Flow speeds at the sections I and III are $u_1+ c$ and $u_3 + c$, respectively.

For the case of the hydraulic jump problem, $u_1$and h_1 are known numbers, and u_3 and h_3 are unknown numbers. In the case of a bore, u_1, h_1, and h_3 are the known numbers, and u_3 and the bore speed c are unknowns. The equation (10) for the jump problem is also satisfied for the bore problem and so

$$q^2 = \frac{gh_1h_3(h_1+h_3)}{2} \qquad (20)$$

is satisfied also for the bore problem. Total flow amount q is "appeared flow by a driving observer," and thus,

$$q = (u_1+ c)\, h_1 = (u_3+ c)\, h_3 \qquad (21)$$

Together with (23) and (24), we can obtain the following:

$$c = \sqrt{\frac{gh_3(h_1+h_3)}{2h_1}} - u_1 \qquad (22)$$

and

$$u_3 = \frac{(u_1+c)h_1}{h_3} - c \qquad (23)$$